# ON THE COMPATIBILITY EQUATIONS IN TERMS OF STRAINS AND STRESSES

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In a number of cases it is convenient to solve problems of the theory of elasticity in terms of stresses. In order to do this, it is necessary to append the equations of compatibility in terms of stresses to the equations of motion. These compatibility equations follow from the equations of compatibility in terms of strain and the constitutive equations – the stress-strain relations. For media which are physically and geometrically linear, the compatibility equations in terms of strains are the Saint-Venant compatibility equations, and those in terms of stresses are the Beltrami-Michell equations. The generalizations of these equations for geometrically nonlinear media are developed below.

**1. A property of fourth-order tensors of special form.** Let us consider a three-dimensional space in which the distance between any two nearby points is given by the positive definite quadratic form

$$ds^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta \tag{1.1}$$

where the functions $g_{\alpha\beta}(x^1, x^2, x^3)$ are the components of the symmetric metric tensor. Here and in what follows it will be assumed that a summation is carried out with respect to a script which occurs twice and that the Greek scripts take on the values 1, 2, 3.

Let us take a fourth-order tensor $A_{\nu\varkappa\lambda\mu}$ in this space having the following symmetries

$$A_{\varkappa\nu\lambda\mu} = -A_{\nu\varkappa\lambda\mu}, \quad A_{\nu\varkappa\mu\lambda} = -A_{\nu\varkappa\lambda\mu}, \quad A_{\lambda\mu\nu\varkappa} = A_{\nu\varkappa\lambda\mu} \tag{1.2}$$

As is easy to show, in three-dimensional space this tensor has only six independent components, corresponding, say, to the following values of the scripts:

$$\nu\varkappa\lambda\mu = 2112,\ 3223,\ 1331,\ 3121,\ 1232,\ 2313 \tag{1.3}$$

In general it is possible to associate six contracted tensors of second order with a fourth-order tensor by means of contraction of two scripts. However, for the tensor under consideration, only one of these contracted tensors is independent because of the symmetries (1.2). This one may be taken as that obtained by contraction with respect to the first and last scripts

$$A_{\varkappa\lambda} = g^{\nu\mu} A_{\nu\varkappa\lambda\mu} \tag{1.4}$$

where

$$\| g^{\nu\mu} \| = \| g_{\nu\mu} \|^{-1}, \qquad g^{\nu\sigma} g_{\sigma\mu} = \delta_\mu^{\ \nu} = \begin{cases} 1, & \nu = \mu \\ 0, & \nu \neq \mu \end{cases} \tag{1.5}$$

The contraction (1.4) is a symmetric tensor. This assertion follows from the symmetry properties of the tensors occurring in its definition. Indeed,

$$A_{\lambda\varkappa} = g^{\nu\mu} A_{\nu\lambda\varkappa\mu} = g^{\nu\mu} A_{\varkappa\mu\nu\lambda} = g^{\mu\nu} A_{\mu\varkappa\lambda\nu} = A_{\varkappa\lambda}$$

Therefore, independent equations are determined in (1.4) for the values of the scripts

$$\varkappa\lambda = 11,\ 22,\ 33,\ 12,\ 23,\ 31 \tag{1.6}$$

Thus, in a three-dimensional space the fourth-order tensor $A_{\nu\varkappa\lambda\mu}$ with the symmetries (1.2) has exactly the same number of independent components as the symmetric tensor $A_{\varkappa\lambda}$. The tensor $A_{\varkappa\lambda}$ is expressed in terms of the tensor $A_{\nu\varkappa\lambda\mu}$ by Eq. (1.4). We shall show that, conversely, $A_{\nu\varkappa\lambda\mu}$ can be expressed in terms of the symmetric tensor $A_{\varkappa\lambda}$. To do this we form a fourth-order tensor having the symmetries (1.2) from some symmetric second-order tensor $B_{\varkappa\lambda}$ and the metric tensor $g_{\nu\mu}$. We now examine the equations

$$A_{\nu\varkappa\lambda\mu}=g_{\varkappa\lambda}B_{\nu\mu}-g_{\varkappa\mu}B_{\nu\lambda}+g_{\nu\mu}B_{\varkappa\lambda}-g_{\nu\lambda}B_{\varkappa\mu} \tag{1.7}$$

in which the scripts take on the values (1.3). These equations may be regarded as a system of six linear, algebraic equations in the six unknown quantities $B_{\alpha\beta}$ $(\alpha\leqslant\beta=1,2,3)$.

It is not difficult to verify that the determinant of the system (1.7) has the value

$$-2\det\|g_{\alpha\beta}\| \qquad (\alpha,\ \beta=1,\ 2,\ 3)$$

and is then nonzero because of the positive definiteness of the form (1.1). The system of Eqs. (1.7) is, therefore, solvable. In order to compute the solution of this system, we transform it to an equivalent system. It follows from the symmetry properties of the tensors on the right-hand side of (1.7) that the system of six equations under consideration is equivalent to the full system (1.7) in which the scripts $\nu\varkappa\lambda\mu$ may all take on the values 1, 2, 3. Now multiply the equations of the system (1.7) by $g^{\nu\mu}$ and sum on the scripts $\nu$ and $\mu$. Then considering Eqs. (1.4) and (1.5), we find as a result that the system (1.7) can be expressed in the form

$$A_{\varkappa\lambda}=Jg_{\varkappa\lambda}+B_{\varkappa\lambda} \qquad (J=g^{\sigma\tau}B_{\sigma\tau}) \tag{1.8}$$

The following equation is a consequence of (1.8):

$$I\equiv g^{\varkappa\lambda}A_{\varkappa\lambda}=4J \tag{1.9}$$

This relates the invariants of the tensors $A_{\varkappa\lambda}$ and $B_{\varkappa\lambda}$. Equations (1.8) and (1.9) permit us to determine the unknowns in the form

$$B_{\varkappa\lambda}=A_{\varkappa\lambda}-{}^1\!/_4 I g_{\varkappa\lambda}$$

Returning to the relations (1.7), we find the components of the tensor $A_{\nu\varkappa\lambda\mu}$ expressed in terms of the components of $A_{\varkappa\lambda}$ by means of the linear homogeneous equations

$$A_{\nu\varkappa\lambda\mu}=g_{\varkappa\lambda}A_{\nu\mu}-g_{\varkappa\mu}A_{\nu\lambda}+g_{\nu\mu}A_{\varkappa\lambda}-g_{\nu\lambda}A_{\varkappa\mu}+{}^1\!/_2 I\,(g_{\varkappa\mu}g_{\nu\lambda}-g_{\varkappa\lambda}g_{\nu\mu}) \tag{1.10}$$

These equations then solve the problem posed.

The relations (1.4) and (1.10) allow us to assert that if one of the tensors $A_{\varkappa\lambda}$, $A_{\nu\varkappa\lambda\mu}$ is zero, then the other one is also. Thus the following Theorem is proved. A necessary and sufficient condition for the vanishing of a tensor $A_{\nu\varkappa\lambda\mu}$ having the symmetries (1.2) is the vanishing of its contraction $A_{\varkappa\lambda}$.

This Theorem makes it possible, for instance, to express the conditions for a space to be Euclidean in various forms. It is well known [1] that the condition of the space being Euclidean is the vanishing of the Riemann-Christoffel tensor

$$R_{\nu\varkappa\lambda\mu}=\frac{\partial\Gamma_{\mu\lambda\nu}}{\partial x^{\varkappa}}-\frac{\partial\Gamma_{\mu\lambda\varkappa}}{\partial x^{\nu}}+g^{\alpha\omega}\left(\Gamma_{\omega\lambda\varkappa}\Gamma_{\alpha\mu\nu}-\Gamma_{\omega\lambda\nu}\Gamma_{\alpha\mu\varkappa}\right)=0 \tag{1.11}$$

where

$$2\Gamma_{\alpha\beta\gamma}=\frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}}+\frac{\partial g_{\alpha\gamma}}{\partial x^{\beta}}-\frac{\partial g_{\beta\gamma}}{\partial x^{\alpha}}$$

are the Christoffel symbols. The fourth-order Riemann-Christoffel tensor also possesses the symmetries (1, 2). Therefore, in three dimensions it has six independent components corresponding to the values (1.3) of the scripts $\nu\varkappa\lambda\mu$. The condition of the space being Euclidean is, therefore, a system of six differential equations which the six functions $g_{\alpha\beta}$ $(\alpha\leqslant\beta=1,\ 2,\ 3)$ must satisfy.

The symmetric second-order tensor, the so-called Ricci tensor

$$R_{\varkappa\lambda} = g^{\nu\mu} R_{\nu\varkappa\lambda\mu}$$

is associated with the Riemann-Christoffel tensor by contraction on the first and last scripts. By virtue of the theorem which has been proved, the condition that the space is Euclidean can be stated as the vanishing of the Ricci tensor

$$R_{\varkappa\lambda} = g^{\nu\mu}\left[\frac{\partial\Gamma_{\mu\lambda\nu}}{\partial x^{\varkappa}} - \frac{\partial\Gamma_{\mu\lambda\varkappa}}{\partial x^{\nu}} + g^{\alpha\omega}(\Gamma_{\omega\lambda\varkappa}\Gamma_{\alpha\mu\nu} - \Gamma_{\omega\lambda\nu}\Gamma_{\alpha\mu\varkappa})\right] = 0$$

This condition is another system of six equations in the same six functions $g_{\alpha\beta}$. The independent equations correspond to the values of the scripts in (1.6).

**2. The equations of compatibility in terms of strain.** The deformation of a body as it goes from some initial configuration (corresponding to the instant of time $t^{\circ}$) to the final configuration (corresponding to the instant $t$) is described by a symmetric, second-order tensor $\varepsilon$, the strain tensor. The components of this tensor are defined by the equations [1]

$$2\varepsilon_{\alpha\beta}(x^1, x^2, x^3, t) = g_{\alpha\beta}(x^1, x^2, x^3, t) - \overset{\circ}{g}_{\alpha\beta}(x^1, x^2, x^3, t^{\circ}) \tag{2.1}$$

where $g_{\alpha\beta}$ and $g_{\alpha\beta}$ are the components of the metric tensor in the comoving (material) coordinate system $x^1$, $x^2$, $x^3$ at the instants $t^{\circ}$ and $t$, respectively. For continuous deformation the six functions (2.1) satisfy the compatibility equations. These express the fact that the deforming medium is in a three-dimensional Euclidean space. They are obtained in the following way. We shall identify the material coordinate system $x^1$, $x^2$, $x^3$ at time $t$ with the reference system relative to which the motion of the medium is considered. We take the condition that the space is Euclidean in the form of the vanishing of the Riemann-Christoffel tensor. Then at the instants of time $t$ and $t^{\circ}$ we have, respectively, Eqs. (1.11) and the equations

$$R^{\circ}{}_{\nu\varkappa\lambda\mu} = \frac{\partial\Gamma^{\circ}{}_{\mu\lambda\nu}}{\partial x^{\varkappa}} - \frac{\partial\Gamma^{\circ}{}_{\mu\lambda\varkappa}}{\partial x^{\nu}} + g_0^{\alpha\omega}(\Gamma^{\circ}{}_{\omega\lambda\varkappa}\Gamma^{\circ}{}_{\alpha\mu\nu} - \Gamma^{\circ}{}_{\omega\lambda\nu}\Gamma^{\circ}{}_{\alpha\mu\varkappa}) = 0 \tag{2.2}$$

in which the values of the quantities at time $t^{\circ}$ are indicated by the degree symbol. With the aid of the equations

$$g^{\circ}{}_{\alpha\beta} = g_{\alpha\beta} - 2\varepsilon_{\alpha\beta} \tag{2.3}$$

which follow from the definition of the components of the strain tensor, a connection can be established between the Christoffel symbols at the two instants of time in the form

$$\Gamma^{\circ}{}_{\alpha\beta\gamma} = \Gamma_{\alpha\beta\gamma} - G_{\alpha\beta\gamma} \qquad \left(G_{\alpha\beta\gamma} = \frac{\partial\varepsilon_{\alpha\beta}}{\partial x^{\gamma}} + \frac{\partial\varepsilon_{\alpha\gamma}}{\partial x^{\beta}} - \frac{\partial\varepsilon_{\beta\gamma}}{\partial x^{\alpha}}\right) \tag{2.4}$$

Substituting (2.3) and (2.4) into (2.2) and subtracting the result from Eq. (1.11), we obtain

$$\frac{\partial^2\varepsilon_{\mu\nu}}{\partial x^{\varkappa}\partial x^{\lambda}} + \frac{\partial^2\varepsilon_{\lambda\varkappa}}{\partial x^{\nu}\partial x^{\mu}} - \frac{\partial^2\varepsilon_{\lambda\nu}}{\partial x^{\varkappa}\partial x^{\mu}} - \frac{\partial^2\varepsilon_{\mu\varkappa}}{\partial x^{\nu}\partial x^{\lambda}} - (g_0^{\alpha\omega} - g^{\alpha\omega})\,\Gamma_{\omega\lambda\varkappa\alpha\mu\nu} -$$
$$- g_0^{\alpha\omega}(G_{\omega\lambda\varkappa\alpha\mu\nu} - B_{\omega\lambda\varkappa\alpha\mu\nu}) = 0 \tag{2.5}$$

where, in accordance with the values in (1.3), six equations are determined by the scripts $\nu\varkappa\lambda\mu$, and where

$$\Gamma_{\omega\lambda\varkappa\alpha\mu\nu} = \Gamma_{\omega\lambda\varkappa}\Gamma_{\alpha\mu\nu} - \Gamma_{\omega\lambda\nu}\Gamma_{\alpha\mu\varkappa}, \qquad G_{\omega\lambda\varkappa\alpha\mu\nu} = G_{\omega\lambda\varkappa}G_{\alpha\mu\nu} - G_{\omega\lambda\nu}G_{\alpha\mu\varkappa}$$

$$B_{\omega\lambda\varkappa\alpha\mu\nu} = \Gamma_{\omega\lambda\varkappa}G_{\alpha\mu\nu} + G_{\omega\lambda\varkappa}\Gamma_{\alpha\mu\nu} - \Gamma_{\omega\lambda\nu}G_{\alpha\mu\varkappa} - G_{\omega\lambda\nu}\Gamma_{\alpha\mu\varkappa}$$

$$\| g_0^{\alpha\omega} \| = \| \overset{\circ}{g}_{\alpha\omega} \|^{-1} = \| g_{\alpha\omega} - 2\varepsilon_{\alpha\omega} \|^{-1}$$

These are then the equations of compatibility written in arbitrary coordinates. We shall show that the left-hand sides of these equations can be expressed in terms of the covariant components of the strain tensor and the covariant derivatives of these components. Let us take the first and second covariant derivatives of the components of the strain tensor

$$\nabla_\lambda \varepsilon_{\mu\nu} = \frac{\partial \varepsilon_{\mu\nu}}{\partial x^\lambda} - \varepsilon_{\sigma\nu}\Gamma^\sigma_{\mu\lambda} - \varepsilon_{\mu\sigma}\Gamma^\sigma_{\nu\lambda}, \quad \Gamma^\sigma_{\mu\lambda} = g^{\sigma\tau}\Gamma_{\tau\mu\lambda}$$

$$\nabla_\varkappa \nabla_\lambda \varepsilon_{\mu\nu} = \frac{\partial}{\partial x^\varkappa}\nabla_\lambda \varepsilon_{\mu\nu} - \nabla_\tau \varepsilon_{\mu\nu}\Gamma^\tau_{\lambda\varkappa} - \nabla_\lambda \varepsilon_{\tau\nu}\Gamma^\tau_{\mu\varkappa} - \nabla_\lambda \varepsilon_{\mu\tau}\Gamma^\tau_{\nu\varkappa}$$

It is easy to verify that the following relations hold in Euclidean space:

$$\nabla_\varkappa\nabla_\lambda \varepsilon_{\mu\nu} + \nabla_\nu\nabla_\mu \varepsilon_{\lambda\varkappa} - \nabla_\varkappa\nabla_\mu \varepsilon_{\lambda\nu} - \nabla_\nu\nabla_\lambda \varepsilon_{\mu\varkappa} = \frac{\partial^2 \varepsilon_{\mu\nu}}{\partial x^\varkappa \partial x^\lambda} + \frac{\partial^2 \varepsilon_{\lambda\varkappa}}{\partial x^\nu \partial x^\mu} -$$

$$- \frac{\partial^2 \varepsilon_{\lambda\nu}}{\partial x^\varkappa \partial x^\mu} - \frac{\partial^2 \varepsilon_{\mu\varkappa}}{\partial x^\nu \partial x^\lambda} + g^{\alpha\omega}B_{\omega\lambda\varkappa\alpha\mu\nu} - 2\varepsilon^{\alpha\omega}\Gamma_{\omega\lambda\varkappa\alpha\mu\nu}, \qquad \varepsilon^{\alpha\omega} = g^{\alpha\sigma}g^{\tau\omega}\varepsilon_{\sigma\tau} \tag{2.6}$$

A direct computation will verify the correctness of the equation

$$C_{\omega\lambda\varkappa} \equiv \nabla_\varkappa \varepsilon_{\omega\lambda} + \nabla_\lambda \varepsilon_{\omega\varkappa} - \nabla_\omega \varepsilon_{\lambda\varkappa} = G_{\omega\lambda\varkappa} - 2\varepsilon_\omega{}^\sigma\Gamma_{\sigma\lambda\varkappa} \tag{2.7}$$

$$\varepsilon_\omega{}^\sigma = g^{\sigma\tau}\varepsilon_{\tau\omega} = {}^1\!/_2\,(\delta_\omega{}^\sigma - g^{\sigma\tau}\overset{\circ}{g}_{\tau\omega})$$

On the basis of Eqs. (2.7) and the equation

$$2\overset{\circ}{g}{}^{\alpha\omega}\varepsilon_\omega{}^\sigma = \overset{\circ}{g}{}^{\alpha\sigma} - g^{\alpha\sigma}$$

the following relation can be established:

$$\overset{\circ}{g}{}^{\alpha\omega}(C_{\omega\lambda\varkappa}C_{\alpha\mu\nu} - C_{\omega\lambda\nu}C_{\alpha\mu\varkappa}) = \overset{\circ}{g}{}^{\alpha\omega}G_{\omega\lambda\varkappa\alpha\mu\nu} - (\overset{\circ}{g}{}^{\alpha\omega} - g^{\alpha\omega})(B_{\omega\lambda\varkappa\alpha\mu\nu} -$$

$$- \Gamma_{\omega\lambda\varkappa\alpha\mu\nu}) - 2\varepsilon^{\alpha\omega}\Gamma_{\omega\lambda\varkappa\alpha\mu\nu} \tag{2.8}$$

If (2.8) is subtracted from (2.6), an equation is obtained whose right-hand side coincides with the left-hand side of the equations of compatibility (2.5). Therefore, the equations of compatibility of strain may be written in the form

$$\nabla_\varkappa\nabla_\lambda \varepsilon_{\mu\nu} + \nabla_\nu\nabla_\mu \varepsilon_{\lambda\varkappa} - \nabla_\varkappa\nabla_\mu \varepsilon_{\lambda\nu} - \nabla_\nu\nabla_\lambda \varepsilon_{\mu\varkappa} - \overset{\circ}{g}{}^{\alpha\omega}(C_{\omega\lambda\varkappa}C_{\alpha\mu\nu} - C_{\omega\lambda\nu}C_{\alpha\mu\varkappa}) = 0 \tag{2.9}$$

The quantities $\overset{\circ}{g}_{\alpha\beta}$ are the components of the metric tensor $G_0$ in the material coordinate system at the initial instant of time. However, at the general time $t$ the metric tensor $G$ has the components $g_{\alpha\beta}$ in the material coordinate system, and the quantities $\overset{\circ}{g}_{\alpha\beta}$ at that instant are the components of some other tensor $T$ which differs from the metric tensor. It is easily seen that the quantities $\overset{\circ}{g}{}^{\alpha\beta}$ at the general time $t$ are the components of the tensor $T^{-1}$, the inverse of the tensor $T$. From the Cayley-Hamilton identity for the tensor $T$ taken in the form

$$T\frac{1}{I_3}(T^2 - I_1T + I_2G) = G$$

where $I_1$, $I_2$, $I_3$ are the principal invariants of the tensor $T$, it follows that $T^{-1}$ can be represented in the form

$$T^{-1} = \frac{I_2}{I_3}G - \frac{I_1}{I_3}T + \frac{1}{I_3}T^2$$

In view of Eq. (2.3), the tensor $T$ is a function of the strain tensor

$$T = G - 2\varepsilon$$

A simple calculation shows that the invariants of this tensor are expressed in terms of the principal invariants $J_\alpha$ ($\alpha = 1, 2, 3$) of the strain tensor according to the formulas

$$I_1 = 3 - 2J_1, \quad I_2 = 3 - 4J_1 + 4J_2, \quad I_3 = 1 - 2J_1 + 4J_2 - 8J_3$$

In an arbitrary curvilinear coordinate system the invariants of the strain tensor are expressed in terms of its covariant components as

$$J_1=g^{\alpha\beta}\varepsilon_{\alpha\beta},\quad J_2={}^1/_2e^{\alpha\beta\gamma}e^{\omega\sigma\tau}g_{\alpha\omega}\varepsilon_{\beta\sigma}\varepsilon_{\gamma\tau},\quad J_3={}^1/_6e^{\alpha\beta\gamma}e^{\omega\sigma\tau}\varepsilon_{\alpha\omega}\varepsilon_{\beta\sigma}\varepsilon_{\gamma\tau}$$

where $e^{\alpha\beta\gamma}$ are the components of the alternating tensor.

The inverse tensor is expressed in terms of the strain tensor according to the equation

$$T^{-1}=K_1G+K_2\varepsilon+K_3\varepsilon^2,\qquad K_1=(1-2J_1+4J_2)\Delta$$

$$K_2=2(1-2J_1)\Delta\qquad K_3=4\Delta,\qquad \Delta=(1-2J_1+4I_2-8I_3)^{-1} \tag{2.10}$$

In tensor notation this equation has the form

$$g_0^{\alpha\omega}=K_1g^{\alpha\omega}+K_2g^{\alpha\sigma}g^{\tau\omega}\varepsilon_{\sigma\tau}+K_3g^{\alpha\sigma}g^{\tau\beta}g^{\gamma\omega}\varepsilon_{\sigma\tau}\varepsilon_{\beta\gamma}$$

The relations (2.7) and (2.10) make it possible to express the compatibility equations (2.9) in the form

$$\begin{aligned}&\nabla_\varkappa\nabla_\lambda\varepsilon_{\mu\nu}+\nabla_\nu\nabla_\mu\varepsilon_{\lambda\varkappa}-\nabla_\varkappa\nabla_\mu\varepsilon_{\lambda\nu}-\nabla_\nu\nabla_\lambda\varepsilon_{\mu\varkappa}-(K_1g^{\alpha\omega}+K_2g^{\alpha\sigma}g^{\tau\omega}\varepsilon_{\sigma\tau}+\\&+K_3g^{\alpha\sigma}g^{\tau\beta}g^{\gamma\omega}\varepsilon_{\sigma\tau}\varepsilon_{\beta\gamma})\,[(\nabla_\varkappa\varepsilon_{\omega\lambda}+\nabla_\lambda\varepsilon_{\omega\varkappa}-\nabla_\omega\varepsilon_{\lambda\varkappa})(\nabla_\nu\varepsilon_{\alpha\mu}+\nabla_\mu\varepsilon_{\alpha\nu}-\nabla_\alpha\varepsilon_{\mu\nu})-\\&-(\nabla_\nu\varepsilon_{\omega\lambda}+\nabla_\lambda\varepsilon_{\omega\nu}-\nabla_\omega\varepsilon_{\lambda\nu})(\nabla_\varkappa\varepsilon_{\alpha\mu}+\nabla_\mu\varepsilon_{\alpha\varkappa}-\nabla_\alpha\varepsilon_{\mu\varkappa})]=0\end{aligned} \tag{2.11}$$

which contain only the covariant components of the strain tensor and their first and second covariant derivatives. In these equations the scripts $\nu\varkappa\lambda\mu$ take on the values in (1.3).

Equations (2.11) hold in arbitrary curvilinear coordinates. In the special case of a rectangular Cartesian coordinate system

$$\nabla_\alpha\varepsilon_{\beta\gamma}=\frac{\partial\varepsilon_{\beta\gamma}}{\partial x^\alpha},\qquad g^{\alpha\beta}=\delta_\beta^\alpha$$

the compatibility equations (2.11) take the form

$$\begin{aligned}&\frac{\partial^2\varepsilon_{\mu\nu}}{\partial x^\varkappa\partial x^\lambda}+\frac{\partial^2\varepsilon_{\lambda\varkappa}}{\partial x^\nu\partial x^\mu}-\frac{\partial^2\varepsilon_{\lambda\nu}}{\partial x^\varkappa\partial x^\mu}-\frac{\partial^2\varepsilon_{\mu\varkappa}}{\partial x^\nu\partial x^\lambda}-(K_1\delta_{\alpha\omega}+K_2\varepsilon_{\alpha\omega}+K_3\varepsilon_{\alpha\sigma}\varepsilon_{\sigma\omega})\times\\&\times\left[\left(\frac{\partial\varepsilon_{\omega\lambda}}{\partial x^\varkappa}+\frac{\partial\varepsilon_{\omega\varkappa}}{\partial x^\lambda}-\frac{\partial\varepsilon_{\lambda\varkappa}}{\partial x^\omega}\right)\left(\frac{\partial\varepsilon_{\alpha\mu}}{\partial x^\nu}+\frac{\partial\varepsilon_{\alpha\nu}}{\partial x^\mu}-\frac{\partial\varepsilon_{\mu\nu}}{\partial x^\alpha}\right)-\right.\\&\left.-\left(\frac{\partial\varepsilon_{\omega\lambda}}{\partial x^\nu}+\frac{\partial\varepsilon_{\omega\nu}}{\partial x^\lambda}-\frac{\partial\varepsilon_{\lambda\nu}}{\partial x^\omega}\right)\left(\frac{\partial\varepsilon_{\alpha\mu}}{\partial x^\varkappa}+\frac{\partial\varepsilon_{\alpha\varkappa}}{\partial x^\mu}-\frac{\partial\varepsilon_{\mu\varkappa}}{\partial x^\alpha}\right)\right]=0\end{aligned}$$

where the scripts assume the system of values (1.3).

We note that Eqs. (2.11) are valid for arbitrary strains. If the strains are small, so that both the components of the strain tensor themselves and their first and second derivatives with respect to the coordinates are small relative to unity, then the nonlinear terms in (2.11) become small terms of higher order and can be neglected. The compatibility equations can then be written in the form

$$\nabla_\varkappa\nabla_\lambda\varepsilon_{\mu\nu}+\nabla_\nu\nabla_\mu\varepsilon_{\lambda\varkappa}-\nabla_\varkappa\nabla_\mu\varepsilon_{\lambda\nu}-\nabla_\nu\nabla_\lambda\varepsilon_{\mu\varkappa}=0 \tag{2.12}$$

where the scripts take the values of (1.3). These are the Saint-Venant compatibility equations for small strains written in arbitrary coordinates [2]. In rectangular Cartesian coordinates these equations have the form

$$\frac{\partial^2\varepsilon_{\mu\nu}}{\partial x^\varkappa\partial x^\lambda}+\frac{\partial^2\varepsilon_{\lambda\varkappa}}{\partial x^\nu\partial x^\mu}-\frac{\partial^2\varepsilon_{\lambda\nu}}{\partial x^\varkappa\partial x^\mu}-\frac{\partial^2\varepsilon_{\mu\varkappa}}{\partial x^\nu\partial x^\lambda}=0$$

We note that the form of the compatibility equations (2.9) can also be obtained from other considerations. Let us first consider some relations which will be needed. The elements of the coordinate bases of the material coordinate system at the initial and general times are defined in terms of the radius vector of the point at the appropriate time by the equations
$$\mathfrak{a}_{\rho}^{(0)} = \frac{\partial \mathbf{r}_0}{\partial x^{\rho}}, \qquad \mathfrak{a}_{\rho} = \frac{\partial \mathbf{r}}{\partial x^{\rho}}$$

By differentiating with respect to the coordinate $x^{\rho}$ the equation
$$\mathbf{r} - \mathbf{r}_0 = \mathbf{w} = w^{\sigma}\mathfrak{a}_{\sigma} = w_0{}^{\tau}\mathfrak{a}_{\tau}^{(0)}$$
which defines the displacement vector, we obtain equations which relate the base vectors for the different times as follows:
$$\mathfrak{a}_{\rho} = C^{(0)}{}_{\rho\cdot}^{\gamma}\mathfrak{a}_{\gamma}^{(0)} = C_{\rho\beta}^{(0)}\mathfrak{a}_0{}^{\beta}, \qquad \mathfrak{a}_{\omega}^{(0)} = C_{\omega\cdot}{}^{\lambda}\mathfrak{a}_{\lambda} = C_{\omega\mu}\mathfrak{a}^{\mu} \tag{2.13}$$

The coefficients in these expansions are expressed in terms of the components of the displacement vector in the different bases by the formulas
$$C^{(0)}{}_{\rho\cdot}^{\gamma} = \delta_{\rho\cdot}^{\gamma} + \nabla_{\rho}^{(0)}w_0{}^{\gamma}, \qquad C_{\rho\beta}^{(0)} = C^{(0)}{}_{\rho\cdot}^{\gamma}g_{\gamma\beta}^{(0)} = g_{\rho\beta}^{(0)} + \nabla_{\rho}^{(0)}w_{\beta}^{(0)}$$
$$C_{\omega\cdot}{}^{\lambda} = \delta_{\omega\cdot}^{\lambda} - \nabla_{\omega}w^{\lambda}, \qquad C_{\omega\mu} = C_{\omega\cdot}{}^{\lambda}g_{\lambda\mu} = g_{\omega\mu} - \nabla_{\omega}w_{\mu}$$
and are connected by the obvious relations
$$C^{(0)}{}_{\rho\cdot}^{\gamma}C_{\gamma\cdot}^{\beta} = \delta_{\rho\cdot}^{\beta}, \qquad C_{\omega\cdot}^{\gamma}C^{(0)}{}_{\gamma\cdot}^{\beta} = \delta_{\omega\cdot}^{\beta} \tag{2.14}$$

The following Lemma holds: the matrices $C_{\rho\omega}^{(0)}$ and $C_{\rho\omega}$ of the expansions (2.13) are the transposes of each other, i.e.
$$C_{\omega\rho} = C_{\rho\omega}^{(0)} \tag{2.15}$$

To prove the Lemma, we form the scalar product of the vector equations (2.13). We have
$$C_{\omega\mu}\mathfrak{a}^{\mu}\cdot\mathfrak{a}_{\rho} = C_{\rho\beta}^{(0)}\mathfrak{a}_0{}^{\beta}\cdot\mathfrak{a}_{\omega}^{(0)}$$

From this result and the reciprocity of the base vectors
$$\mathfrak{a}^{\mu}\cdot\mathfrak{a}_{\rho} = \delta_{\cdot\rho}^{\mu}, \qquad \mathfrak{a}_0{}^{\beta}\cdot\mathfrak{a}_{\omega}^{(0)} = \delta_{\cdot\omega}^{\beta}$$
we arrive at Eq. (2.15). Lemma is thus proved.

The relation (2.15) can be expressed in another, equivalent form. To this end, we multiply both sides of it by the quantity $g_0^{\alpha\omega}\ \ g^{\sigma\rho}$ and sum on the scripts $\omega$ and $\rho$; as a result we obtain
$$g_0{}^{\alpha\omega}C_{\omega\cdot}^{\sigma} = g^{\sigma\rho}C^{(0)}{}_{\rho\cdot}^{\alpha} \tag{2.16}$$

This means that the matrices in the expansions of the base vectors
$$\mathfrak{a}_0{}^{\alpha} = g_0{}^{\alpha\omega}\mathfrak{a}_{\omega}^{(0)} = g_0{}^{\alpha\omega}C_{\omega\cdot}^{\sigma}\mathfrak{a}_{\sigma}, \qquad \mathfrak{a}_{\cdot}^{\sigma} = g^{\sigma\rho}\mathfrak{a}_{\rho} = g^{\sigma\rho}C^{(0)}{}_{\rho\cdot}^{\alpha}\mathfrak{a}_{\alpha}^{(0)}$$
are also the transposes of each other. In view of the properties (2.14), Eq. (2.16) can also be written in the form
$$g^{\sigma\tau} = g_0{}^{\alpha\omega}C_{\omega\cdot}^{\sigma}C_{\alpha\cdot}^{\tau} \tag{2.17}$$

Let us now return to the derivation of the compatibility equations. It is well known from [1] that the strains are expressible in terms of the displacements by the equations
$$2\varepsilon_{\mu\nu} = \nabla_{\mu}w_{\nu} + \nabla_{\nu}w_{\mu} - g^{\sigma\tau}\nabla_{\mu}w_{\sigma}\nabla_{\nu}w_{\tau} \tag{2.18}$$

The compatibility equations will be obtained if the displacements are eliminated from these equations. In order to do this, we calculate the first and second covariant derivatives of the components of the strain tensor with respect to the coordinates

$$2\nabla_\lambda \varepsilon_{\mu\nu} = \nabla_\lambda \nabla_\mu w_\nu + \nabla_\lambda \nabla_\nu w_\mu - g^{\sigma\tau}(\nabla_\lambda \nabla_\mu w_\sigma \nabla_\nu w_\tau + \nabla_\mu w_\sigma \nabla_\lambda \nabla_\nu w_\tau) \tag{2.19}$$

$$2\nabla_\varkappa \nabla_\lambda \varepsilon_{\mu\nu} = \nabla_\varkappa \nabla_\lambda \nabla_\mu w_\nu + \nabla_\varkappa \nabla_\lambda \nabla_\nu w_\mu - \tag{2.20}$$
$$- g^{\sigma\tau}(\nabla_\varkappa \nabla_\lambda \nabla_\mu w_\sigma \nabla_\nu w_\tau + \nabla_\lambda \nabla_\mu w_\sigma \nabla_\varkappa \nabla_\nu w_\tau + \nabla_\varkappa \nabla_\mu w_\sigma \nabla_\lambda \nabla_\nu w_\tau + \nabla_\mu w_\sigma \nabla_\varkappa \nabla_\lambda \nabla_\nu w_\tau)$$

As is easily verified, it is possible to eliminate the third derivatives of the displacements from (2.20) by forming the following combination:

$$\nabla_\varkappa \nabla_\lambda \varepsilon_{\mu\nu} + \nabla_\nu \nabla_\mu \varepsilon_{\lambda\varkappa} - \nabla_\varkappa \nabla_\mu \varepsilon_{\lambda\nu} - \nabla_\nu \nabla_\lambda \varepsilon_{\mu\varkappa} = \tag{2.21}$$
$$= g^{\sigma\tau}(\nabla_\nu \nabla_\mu w_\sigma \nabla_\varkappa \nabla_\lambda w_\tau - \nabla_\lambda \nabla_\mu w_\sigma \nabla_\varkappa \nabla_\nu w_\tau)$$

The expression which is obtained contains only second derivatives of the displacements. It turns out to be possible to eliminate the latter using Eqs. (2.19). In fact

$$C_{\omega\lambda\varkappa} \equiv \nabla_\varkappa \varepsilon_{\omega\lambda} + \nabla_\lambda \varepsilon_{\omega\varkappa} - \nabla_\omega \varepsilon_{\lambda\varkappa} = \nabla_\varkappa \nabla_\lambda w_\sigma C_\omega{}^\sigma .$$

Therefore,

$$C_{\omega\lambda\varkappa} C_{\alpha\mu\nu} - C_{\omega\lambda\nu} C_{\alpha\mu\varkappa} = C_{\omega}{}^{\sigma} C_{\alpha}{}^{\tau} (\nabla_\nu \nabla_\mu w_\sigma \nabla_\varkappa \nabla_\lambda w_\tau - \nabla_\lambda \nabla_\mu w_\sigma \nabla_\varkappa \nabla_\nu w_\tau)$$

We then multiply both sides of the last equation by $g_0^{\alpha\omega}$, sum on $\alpha$ and $\omega$, and by use of the relation (2.17) we have

$$g_0{}^{\alpha\omega}(C_{\omega\lambda\varkappa} C_{\alpha\mu\nu} - C_{\omega\lambda\nu} C_{\alpha\mu\varkappa}) = g^{\sigma\tau}(\nabla_\nu \nabla_\mu w_\sigma \nabla_\varkappa \nabla_\lambda w_\tau - \nabla_\lambda \nabla_\mu w_\sigma \nabla_\varkappa \nabla_\nu w_\tau) \tag{2.22}$$

It is now easy to see that by subtracting (2.22) from (2.21) we eliminate the displacements entirely and thereby obtain equations for the quantities $\varepsilon_{\alpha\beta}$ which coincide with the compatibility equations in the form (2.9).

Thus the expressions (2.18) for the strains in terms of the displacements may be regarded as integrals of the equations of compatibility.

We remark that for strains and angles of rotation which are small compared with unity, the nonlinear equations (2.18) may be replaced by the linear relations [3]

$$2\varepsilon_{\mu\nu} = \nabla_\mu w_\nu + \nabla_\nu w_\mu \tag{2.23}$$

Computing the second covariant derivatives of these and forming the combination (2.21) we obtain equations which do not contain the displacements and which agree with the Saint-Venant equations (2.12) Therefore, the Saint-Venant equations are a consequence of the linear equations (2.23).

The equations of compatibility state that a certain fourth-order tensor is zero. It is not difficult to verify that this tensor possesses the symmetries (1.2). Therefore, by virtue of the Theorem of Sect. 1, Eqs. (2.9) are equivalent to the equations obtained from them by contracting with respect to the scripts $\nu$ and $\mu$. These equations have the form

$$\Delta\varepsilon_{\varkappa\lambda} + \nabla_\varkappa \nabla_\lambda J_1 - \nabla_\varkappa \nabla^\mu \varepsilon_{\lambda\mu} - \nabla_\lambda \nabla^\mu \varepsilon_{\varkappa\mu} - g_0^{\alpha\omega} g^{\mu\nu}(C_{\omega\lambda\varkappa} C_{\alpha\mu\nu} - C_{\omega\lambda\nu} C_{\alpha\mu\varkappa}) = 0 \tag{2.24}$$

where $\nabla^\mu$ is the operator of contravariant differentiation, $\Delta$ is the Laplacian operator, and the scripts $\varkappa\lambda$ have the values of (1.6). Written out in full, Eqs. (2.24) have the expressions

$$\Delta\varepsilon_{\varkappa\lambda} + \nabla_\varkappa \nabla_\lambda J_1 - \nabla_\varkappa \nabla^\mu \varepsilon_{\lambda\mu} - \nabla_\lambda \nabla^\mu \varepsilon_{\varkappa\mu} - (K_1 g^{\alpha\omega} + K_2 g^{\alpha\sigma} g^{\tau\omega} \varepsilon_{\sigma\tau} +$$
$$+ K_3 g^{\alpha\sigma} g^{\tau\beta} g^{\gamma\omega} \varepsilon_{\sigma\tau} \varepsilon_{\beta\gamma})[(\nabla_\varkappa \varepsilon_{\omega\lambda} + \nabla_\lambda \varepsilon_{\omega\varkappa} - \nabla_\omega \varepsilon_{\lambda\varkappa})(2\nabla^\mu \varepsilon_{\alpha\mu} - \nabla_\alpha J_1) -$$
$$- g^{\mu\nu}(\nabla_\nu \varepsilon_{\omega\lambda} + \nabla_\lambda \varepsilon_{\omega\nu} - \nabla_\omega \varepsilon_{\lambda\nu})(\nabla_\varkappa \varepsilon_{\alpha\mu} + \nabla_\mu \varepsilon_{\alpha\varkappa} - \nabla_\alpha \varepsilon_{\mu\varkappa})] = 0$$

The equations which have been obtained are a special form of the compatibility equations; they are equivalent to Eqs. (2.11). In rectangular Cartesian coordinates these equations are of the form

$$\frac{\partial^2 \varepsilon_{\varkappa\lambda}}{\partial x^\mu \partial x^\mu} + \frac{\partial^2 J_1}{\partial x^\varkappa \partial x^\lambda} - \frac{\partial^2 \varepsilon_{\lambda\mu}}{\partial x^\varkappa \partial x^\mu} - \frac{\partial^2 \varepsilon_{\varkappa\mu}}{\partial x^\lambda \partial x^\mu} - (K_1\delta_{\alpha\omega} + K_2\varepsilon_{\alpha\omega} + K_3\varepsilon_{\alpha\sigma}\varepsilon_{\sigma\omega}) \times$$
$$\times \left[\left(\frac{\partial \varepsilon_{\omega\lambda}}{\partial x^\varkappa} + \frac{\partial \varepsilon_{\omega\varkappa}}{\partial x^\lambda} - \frac{\partial \varepsilon_{\lambda\varkappa}}{\partial x^\omega}\right)\left(2\frac{\partial \varepsilon_{\alpha\mu}}{\partial x^\mu} - \frac{\partial J_1}{\partial x^\alpha}\right) - \right.$$
$$\left. - \left(\frac{\partial \varepsilon_{\omega\lambda}}{\partial x^\mu} + \frac{\partial \varepsilon_{\omega\mu}}{\partial x^\lambda} - \frac{\partial \varepsilon_{\lambda\mu}}{\partial x^\omega}\right)\left(\frac{\partial \varepsilon_{\alpha\mu}}{\partial x^\varkappa} + \frac{\partial \varepsilon_{\alpha\varkappa}}{\partial x^\mu} - \frac{\partial \varepsilon_{\mu\varkappa}}{\partial x^\alpha}\right)\right] = 0$$

For small deformations it is permissible to neglect the nonlinear terms in (2.24) and to write the equations in the form

$$\Delta\varepsilon_{\varkappa\lambda} + \nabla_\varkappa\nabla_\lambda J_1 - \nabla_\varkappa\nabla^\mu\varepsilon_{\lambda\mu} - \nabla_\lambda\nabla^\mu\varepsilon_{\varkappa\mu} = 0 \tag{2.25}$$

This is a special form of the Saint-Venant equations of compatibility for small strains given in arbitrary coordinates [4]. In Cartesian coordinates they have the form

$$\frac{\partial^2 \varepsilon_{\varkappa\lambda}}{\partial x^\mu \partial x^\mu} + \frac{\partial^2 J_1}{\partial x^\varkappa \partial x^\lambda} - \frac{\partial^2 \varepsilon_{\lambda\mu}}{\partial x^\varkappa \partial x^\mu} - \frac{\partial^2 \varepsilon_{\varkappa\mu}}{\partial x^\lambda \partial x^\mu} = 0$$

**3. The compatibility equations in terms of stresses.** In the physically linear theory of elasticity it is assumed that the constitutive law for the medium is the generalized Hooke's law, which states that the components of the strain tensor $\varepsilon_{\alpha\beta}$ are homogeneous linear functions of the stress tensor $P_{\rho\tau}$

$$\varepsilon_{\alpha\beta} = \frac{1+\nu}{E} P_{\alpha\beta} - \frac{\nu}{E} J_1 g_{\alpha\beta} \tag{3.1}$$

where $J_1$ is the first invariant of the stress tensor, $E$ is Young's modulus, and $\nu$ is Poisson's ratio.

Since for continuous deformation of a medium the components of the strain tensor satisfy the equations of compatibility, it follows from Hooke's law that the components of the stress tensor also satisfy certain equations if the deformation is to be continuous. These equations are called the equations of compatibility in terms of stresses. Let us obtain explicit expressions for them.

The principal invariants of the stress tensor are defined by the equations

$$J_1 = g^{\alpha\beta}P_{\alpha\beta}, \quad J_2 = {}^1/_2 e^{\alpha\beta\gamma}e^{\omega\sigma\tau}g_{\alpha\omega}P_{\beta\sigma}P_{\gamma\tau}, \quad J_3 = {}^1/_6 e^{\alpha\beta\gamma}e^{\omega\sigma\tau}P_{\alpha\omega}P_{\beta\sigma}P_{\gamma\tau}$$

The following relations between the invariants of the strain and stress tensors are consequences of Hooke's law:

$$J_1 = \frac{1-2\nu}{E} J_1, \quad J_2 = \frac{1}{E^2}[(1+\nu)^2 J_2 - \nu(2-\nu)J_1^2]$$
$$J_3 = \frac{1}{E^3}[(1+\nu)^3 J_3 - \nu(1+\nu)^2 J_1 J_2 + \nu^2 J_1^3]$$

Returning to Eqs. (2.10), we see that they may be written as

$$g_0^{\alpha\omega} = L_1 g^{\alpha\omega} + L_2 g^{\alpha\sigma}g^{\tau\omega}P_{\sigma\tau} + L_3 g^{\alpha\sigma}g^{\tau\beta}g^{\gamma\omega}P_{\sigma\tau}P_{\beta\gamma} \tag{3.2}$$
$$L_1 = K_1 - \frac{\nu}{E}J_1K_2 + \frac{\nu^2}{E^2}J_1^2K_3, \quad L_2 = \frac{1+\nu}{E}\left(K_2 - \frac{2\nu}{E}J_1K_3\right), \quad L_3 = \frac{(1+\nu)^2}{E^2}K_3$$

and are clearly functions of the stress invariants. The covariant derivatives of the stress and strain tensors are related as follows:

$$\nabla_\varkappa\varepsilon_{\omega\lambda} = \frac{1+\nu}{E}\nabla_\varkappa P_{\omega\lambda} - \frac{\nu}{E}\nabla_\varkappa J_1 g_{\omega\lambda}$$

$$\nabla_\mu \nabla_\varkappa \varepsilon_{\omega\lambda} = \frac{1+\nu}{E} \nabla_\mu \nabla_\varkappa P_{\omega\lambda} - \frac{\nu}{E} \nabla_\mu \nabla_\varkappa J_1 g_{\omega\lambda}$$

Therefore, we have

$$\Delta \varepsilon_{\varkappa\lambda} = \frac{1+\nu}{E} \Delta P_{\varkappa\lambda} - \frac{\nu}{E} \Delta J_1 g_{\varkappa\lambda}, \quad \nabla_\varkappa \nabla_\lambda J_1 = \frac{1-2\nu}{E} \nabla_\varkappa \nabla_\lambda J_1$$

$$\nabla_\varkappa \nabla^\mu \varepsilon_{\lambda\mu} + \nabla_\lambda \nabla^\mu \varepsilon_{\varkappa\mu} = \frac{1+\nu}{E} (\nabla_\varkappa \nabla^\mu P_{\lambda\mu} + \nabla_\lambda \nabla^\mu P_{\varkappa\mu}) - \frac{2\nu}{E} \nabla_\varkappa \nabla_\lambda J_1 \qquad (3.3)$$

$$C_{\omega\lambda\varkappa} = \frac{1+\nu}{E} D_{\omega\lambda\varkappa} - \frac{\nu}{E} F_{\omega\lambda\varkappa} \qquad (3.4)$$

$$D_{\omega\lambda\varkappa} = \nabla_\varkappa P_{\omega\lambda} + \nabla_\lambda P_{\omega\varkappa} - \nabla_\omega P_{\lambda\varkappa}, \quad F_{\omega\lambda\varkappa} = \nabla_\varkappa J_1 g_{\omega\lambda} + \nabla_\lambda J_1 g_{\omega\varkappa} - \nabla_\omega J_1 g_{\lambda\varkappa} \quad (3.5)$$

Substituting Eqs. (3.3) and (3.4) into Eqs. (2.24), we obtain

$$\Delta P_{\varkappa\lambda} + \frac{1}{1+\nu} \nabla_\varkappa \nabla_\lambda J_1 - \frac{\nu}{1+\nu} \Delta J_1 g_{\varkappa\lambda} - \nabla_\varkappa \nabla^\mu P_{\lambda\mu} - \nabla_\lambda \nabla^\mu P_{\varkappa\mu} -$$
$$- g_0^{\alpha\omega} g^{\mu\rho} \left[ \frac{1+\nu}{E} (D_{\omega\lambda\varkappa} D_{\alpha\mu\rho} - D_{\omega\lambda\rho} D_{\alpha\mu\varkappa}) - \frac{\nu}{E} (D_{\omega\lambda\varkappa} F_{\alpha\mu\rho} + F_{\omega\lambda\varkappa} D_{\alpha\mu\rho} - \right.$$
$$\left. - D_{\omega\lambda\rho} F_{\alpha\mu\varkappa} - F_{\omega\lambda\rho} D_{\alpha\mu\varkappa}) + \frac{\nu^2}{E(1+\nu)} (F_{\omega\lambda\varkappa} F_{\alpha\mu\rho} - F_{\omega\lambda\rho} F_{\alpha\mu\varkappa}) \right] = 0$$

In view of Eqs. (3.2) and (3.5), these equations contain only the components of the stress tensor and their first and second covariant derivatives. These are then the equations of compatibility in terms of stresses which are suitable for geometrically nonlinear media.

Some simplifications of these equations are possible. If we use the equations of equilibrium

$$\nabla^\mu P_{\lambda\mu} = - f_\lambda$$

where $f_\lambda$ are the components of the body force, we have

$$\nabla_\varkappa \nabla^\mu P_{\lambda\mu} + \nabla_\lambda \nabla^\mu P_{\varkappa\mu} = - \nabla_\varkappa f_\lambda - \nabla_\lambda f_\varkappa$$

$$g^{\mu\rho} D_{\alpha\mu\rho} = -2 f_\alpha - \nabla_\alpha J_1, \quad g^{\mu\rho} F_{\alpha\mu\rho} = - \nabla_\alpha J_1$$

Taking account of these equations, we mav write the compatibility equations in terms of stress in the form

$$\Delta P_{\varkappa\lambda} + \frac{1}{1+\nu} \nabla_\varkappa \nabla_\lambda J_1 - \frac{\nu}{1+\nu} \Delta J_1 g_{\varkappa\lambda} + \nabla_\varkappa f_\lambda + \nabla_\lambda f_\varkappa +$$
$$+ (L_1 g^{\alpha\omega} + L_2 g^{\alpha\sigma} g^{\tau\omega} P_{\sigma\tau} + L_3 g^{\alpha\sigma} g^{\tau\beta} g^{\gamma\omega} P_{\sigma\tau} P_{\beta\gamma}) \times$$
$$\times \left\{ \frac{1+\nu}{E} [D_{\omega\lambda\varkappa} (2 f_\alpha + \nabla_\alpha J_1) + g^{\mu\rho} D_{\omega\lambda\rho} D_{\alpha\mu\varkappa}] - \right.$$
$$- \frac{\nu}{E} [D_{\omega\lambda\varkappa} \nabla_\alpha J_1 + F_{\omega\lambda\varkappa} (2 f_\alpha + \nabla_\alpha J_1) + g^{\mu\rho} (D_{\omega\lambda\rho} F_{\alpha\mu\varkappa} + F_{\omega\lambda\rho} D_{\alpha\mu\varkappa})] +$$
$$\left. + \frac{\nu^2}{E(1+\nu)} [F_{\omega\lambda\varkappa} \nabla_\alpha J_1 + g^{\mu\rho} F_{\omega\lambda\rho} F_{\alpha\mu\varkappa}] \right\} = 0 \qquad (3.6)$$

The independent equations of this system are determined by the values of $\varkappa$ and $\lambda$ in (1.6).

We note that Eqs. (3.6) are given in tensor form and contain only covariant components of the stress tensor. However, proceeding from these equations we can obtain others containing only contravariant components of stress. All that is needed to do this is to apply the operation of raising the scripts in Eq. (3.6). As a result, we obtain the equations of compatibility in terms of stresses in the form

$$\Delta P^{\varkappa\lambda} + \frac{1}{1+\nu}\nabla^{\varkappa}\nabla^{\lambda}J_1 - \frac{\nu}{1+\nu}\Delta J_1 g^{\varkappa\lambda} + \nabla^{\varkappa} f^{\lambda} + \nabla^{\lambda} f^{\varkappa} + (L_1 g_{\alpha\omega} + L_2 g_{\alpha\sigma} g_{\tau\omega} P^{\sigma\tau} +$$
$$+ L_3 g_{\alpha\sigma} g_{\tau\beta} g_{\gamma\omega} P^{\sigma\tau} P^{\beta\gamma}) \Big\{ \frac{1+\nu}{E} [D^{\omega\lambda\varkappa}(2f^{\alpha} + \nabla^{\alpha} J_1) + g_{\mu\rho} D^{\omega\lambda\rho} D^{\alpha\mu\varkappa}] -$$
$$- \frac{\nu}{E} [D^{\omega\lambda\varkappa}\nabla^{\alpha} J_1 + F^{\omega\lambda\varkappa}(2f^{\alpha} + \nabla^{\alpha} J_1) + g_{\mu\rho}(D^{\omega\lambda\rho} F^{\alpha\mu\varkappa} + F^{\omega\lambda\rho} D^{\alpha\mu\varkappa})] +$$
$$+ \frac{\nu^2}{E(1+\nu)} [F^{\omega\lambda\varkappa}\nabla^{\alpha} J_1 + g_{\mu\rho} F^{\omega\lambda\rho} F^{\alpha\mu\varkappa}] \Big\} = 0 \qquad (3.7)$$

where

$$P^{\varkappa\lambda} = g^{\varkappa\sigma} g^{\tau\lambda} P_{\sigma\tau}, \qquad f^{\lambda} = g^{\lambda\sigma} f_{\sigma}$$
$$D^{\omega\lambda\varkappa} = \nabla^{\varkappa} P^{\omega\lambda} + \nabla^{\lambda} P^{\omega\varkappa} - \nabla^{\omega} P^{\lambda\varkappa}, \qquad F^{\omega\lambda\varkappa} = \nabla^{\varkappa} J_1 g^{\omega\lambda} + \nabla^{\lambda} J_1 g^{\omega\varkappa} - \nabla^{\omega} J_1 g^{\lambda\varkappa}$$

and the scripts have the values in (1.6).

In the special case of a rectangular Cartesian coordinate system, the forms (3.6) and (3.7) of the compatibility equations in terms of stresses coincide and are

$$\frac{\partial^2 P_{\varkappa\lambda}}{\partial x^{\sigma}\partial x^{\sigma}} + \frac{1}{1+\nu}\frac{\partial^2 J_1}{\partial x^{\varkappa}\partial x^{\lambda}} - \frac{\nu}{1+\nu}\frac{\partial^2 J_1}{\partial x^{\sigma}\partial x^{\sigma}}\delta_{\varkappa\lambda} + \frac{\partial f_{\lambda}}{\partial x^{\varkappa}} + \frac{\partial f_{\varkappa}}{\partial x^{\lambda}} +$$
$$+ (L_1\delta_{\alpha\omega} + L_2 P_{\alpha\omega} + L_3 P_{\alpha\sigma} P_{\sigma\omega}) \Big\{ \frac{1+\nu}{E}\Big[D'_{\omega\lambda\varkappa}\Big(2f_{\alpha} + \frac{\partial J_1}{\partial x^{\alpha}}\Big) + D'_{\omega\lambda\rho} D'_{\alpha\rho\varkappa}\Big] -$$
$$- \frac{\nu}{E}\Big[D'_{\omega\lambda\varkappa}\frac{\partial J_1}{\partial x^{\alpha}} + F'_{\omega\lambda\varkappa}\Big(2f_{\alpha} + \frac{\partial J_1}{\partial x^{\alpha}}\Big) + D'_{\omega\lambda\rho} F'_{\alpha\rho\varkappa} + F'_{\omega\lambda\rho} D'_{\alpha\rho\varkappa}\Big] +$$
$$+ \frac{\nu^2}{E(1+\nu)}\Big[F'_{\omega\lambda\varkappa}\frac{\partial J_1}{\partial x^{\alpha}} + F'_{\omega\lambda\rho} F'_{\alpha\rho\varkappa}\Big]\Big\} = 0$$

where

$$D'_{\omega\lambda\varkappa} = \frac{\partial P_{\omega\lambda}}{\partial x^{\varkappa}} + \frac{\partial P_{\omega\varkappa}}{\partial x^{\lambda}} - \frac{\partial P_{\lambda\varkappa}}{\partial x^{\omega}}, \qquad F'_{\omega\lambda\varkappa} = \frac{\partial J_1}{\partial x^{\varkappa}}\delta_{\omega\lambda} + \frac{\partial J_1}{\partial x^{\lambda}}\delta_{\omega\varkappa} - \frac{\partial J_1}{\partial x^{\omega}}\delta_{\lambda\varkappa}$$

The compatibility equations in terms of stresses (3.6) correspond to the general nonlinear strain-displacement relations (2.18). For small strains and rotations these relations are linear and the compatibility equations (2.25) are also linear. In this case, the compatibility equations in terms of stresses are linear as well. It is easy to see that they are just the linear part of Eqs. (3.6)

$$\Delta P_{\varkappa\lambda} + \frac{1}{1+\nu}\nabla_{\varkappa}\nabla_{\lambda}J_1 - \frac{\nu}{1+\nu}\Delta J_1 g_{\varkappa\lambda} + \nabla_{\varkappa} f_{\lambda} + \nabla_{\lambda} f_{\varkappa} = 0 \qquad (3.8)$$

These equations can be simplified further. In the case under consideration the stresses are expressed in terms of the displacements by

$$P_{\lambda\mu} = \frac{\nu}{1+\nu} J_1 g_{\lambda\mu} + \frac{E}{2(1+\nu)}(\nabla_{\lambda} w_{\mu} + \nabla_{\mu} w_{\lambda})$$

It follows from this that the first invariant of the stresses is proportional to the divergence of the displacements

$$J_1 = \frac{E}{1-2\nu}\nabla_{\alpha} w^{\alpha}$$

From the divergence of the equations of equilibrium

$$\nabla_{\alpha} J_1 + E\Delta w_{\alpha} = -2(1+\nu) f_{\alpha}$$

we can express $\Delta J_1$ in terms of the specified body forces in the form

$$\Delta J_1 = -\frac{1+\nu}{1-\nu}\nabla^{\alpha} f_{\alpha}$$

Taking account of this result, we write (3.8) in the final form

$$\Delta P_{\varkappa\lambda} + \frac{1}{1+\nu} \nabla_\varkappa \nabla_\lambda J_1 + \frac{\nu}{1-\nu} \nabla^\alpha f_\alpha g_{\varkappa\lambda} + \nabla_\varkappa f_\lambda + \nabla_\lambda f_\varkappa = 0 \qquad (3.9)$$

These are the Beltrami-Michell equations of compatibility in terms of stresses which are presented in the linear theory of elasticity [4, 5]. Raising the scripts $\varkappa$ and $\lambda$, we obtain the equations in the contravariant stress components

$$\Delta P^{\varkappa\lambda} + \frac{1}{1+\nu} \nabla^\varkappa \nabla^\lambda J_1 + \frac{\nu}{1-\nu} \nabla^\alpha f_\alpha g^{\varkappa\lambda} + \nabla^\varkappa f^\lambda + \nabla^\lambda f^\varkappa = 0 \qquad (3.10)$$

In Cartesian coordinates, Eqs. (3.9) and (3.10) have the form

$$\frac{\partial^2 P_{\varkappa\lambda}}{\partial x^\alpha \partial x^\alpha} + \frac{1}{1+\nu} \frac{\partial^2 J_1}{\partial x^\varkappa \partial x^\lambda} + \frac{\nu}{1-\nu} \frac{\partial f_\alpha}{\partial x^\alpha} \delta_{\varkappa\lambda} + \frac{\partial f_\lambda}{\partial x^\varkappa} + \frac{\partial f_\varkappa}{\partial x^\lambda} = 0$$

Thus, the Beltrami-Michell equations of compatibility in terms of stresses correspond to geometrically and physically linear elasticity; Eqs. (3.6) are the generalizations of these equations in the case of geometric nonlinearity.

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# ON FUNDAMENTAL SOLUTIONS IN THE THEORY OF CIRCULAR CYLINDRICAL SHELLS

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The problem of a circular cylindrical shell of elastic isotropic material subjected to concentrated loadings is considered. As is known, such a problem in two-dimensional formulation (based on Kirchhoff-Love hypotheses) reduces to the construction of the Green's function for an elliptic equation in the resolution function.

A fundamental solution in closed form has been obtained in [1, 2] for the shallow cylindrical shell equations by using Fourier transforms. A method of the theory of generalized functions [4] was utilized in [3] to construct a fundamental solution of the equations of the theory of shells of positive Gaussian curvature.

Fundamental solutions are constructed below for the most prevalent modifications of the theory of nonshallow circular cylindrical shells [5–8]. In contrast to [1–3], the "classical" method of plane waves and spherical means [9] is utilized which permits, so to